Lecture 6 - January 26

Math Review

Relations, Relational Operations

Announcement

- Lab1 submission due in a week
  + Help: scheduled office hours & TA
  + tutorial videos
  + problems to solve
  + Study along with the Math Review lecture notes.

$r \in S \leftrightarrow T$

set of possible relations on S and T

a single relation on S and T

$S = \{1, 2, 3\}$
$T = \{a, b\}$

$r : S \leftrightarrow T$

$\in$

could be:
$(1, a)$ or
$(2, b)$

$r2 \in \underbrace{\{(1, a), (2, b)\}}_{\in S \leftrightarrow T}$

$r \in S \leftrightarrow T$

r could be:
(1) $\emptyset$
(2) $S \times T$
(3) $\{(1, a), (2, b)\}$

Set of Possible Relations

- Set of possible relations on S and T:
- Dedicated symbol for set of possible relations on S and T:
- Declare that set r is a relation on S and T:

Example: Enumerate **all** relations on {a, b} and {2, 4}.

card: $2^4 = 16$

Hint: How many?

$\{a, b\} \leftrightarrow \{2, 4\}$

$\mathbb{P}(\{a, b\} \times \{2, 4\})$

max relation on the two sets

$\{(a,2), (a,4), (b,2), (b,4)\}$

$\mathbb{P}(\{(a,2), (a,4), (b,2), (b,4)\})$

$= \{$

$\emptyset$, /* relation of size 0 */

$\{(a,2)\}, \{(a,4)\}, \{(b,2)\}, \{(b,4)\}$ /* rel of size 1 */

$\{(a,2), (a,4), (b,2), (b,4)\}$

$\}$ /* relation of size 4 */

$\binom{4}{1} = 4$

(a) relations of size 2 $\binom{4}{2} = \frac{4*3}{2!} = 6$

(b) relations of size 3 $\binom{4}{3} = \frac{4*3*2}{3!} = 4$

$$\binom{n}{r}$$

out of n given elements, how many ways to make a set of card. r

$\{a, b, c\}$

$\{a, b, c\}$
$\{b, c, a\}$
$\{a, c, b\}$

n $n-1$ $n-2$

set of card. r

r terms

$$\frac{\overbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}^{r \text{ terms}}}{r!}$$

$r!$ → disregard duplicates

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

$$\text{Departure} = \{\text{toronto}, \text{montreal}, \text{vancouver}\}$$

$$\text{Destination} = \{\text{beijing}, \text{seoul}, \text{penang}\}$$

$$\text{airline} \in \boxed{\text{Departure} \leftrightarrow \text{Destination}}$$

task: enumerate!

Relational Operations: Domain, Range, Inverse

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$dom(r) = \{a, b, c, d, e, f\}$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$ran(r) = \{1, 2, 3, 4, 5, 6\}$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r\sim = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$

$|r| = |r\sim|$

algebraic properties

Exercise: Relate the domains and ranges of r and its inverse.

(1) $dom(r) = ran(r\sim)$ (2) $ran(r) = dom(r\sim)$

Relational Operations: Image

* $r \in S \leftrightarrow T$
$r[s]$ assumption: $s \subseteq S$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r \in \text{Alphabet} \leftrightarrow \mathbb{Z}$

all lower & upper letters

$r[s]$

a set!

$r[\{a, b\}] = \{r' \mid (d, r') \in r \land d \in \{a, b\}\}$
$= \{1, 2, 4, 5\}$

Exercises

- $\boxed{\text{Image of } \{a, b\} \text{ on } r?}$
- Image of {1, 2} on r?* undefined
- Image of {1, 2} on the inverse of r? → $\{a, b, d, e\}$
- Calculate r's range via an image. → $ran(r) = r[dom(r)]$
- Calculate r's domain via an image. → $dom(r) = r^{\sim}[\boxed{ran(r)}]$ $dom(r^{\sim})$

e.g. $r[\emptyset] = \emptyset$

e.g. $r[\{x, y\}] = \emptyset$

$r[\{1, a\}]$ ✗ undefined

$\underline{\underline{r}} \in S \leftrightarrow T \qquad s \subseteq S$

	domain	range
Restriction	$s \lhd r$	$r \rhd s$
Subtraction	$s \ntriangleleft r$	$r \ntriangleright s$

another relation

$$\boxed{\underline{\underline{ds}} \lhd r} = \{(d, r') \mid (d, r') \in r \land d \in ds\}$$

domain restriction

Relational Operations: Restrictions vs. Subtractions

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$\{a, b\} \lhd r = \{(a,1), (b,2), (a,4), (b,5)\}$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r \rhd \{1, 2\} = \{(a,1), (b,2), (d,1), (e,2)\}$

$r = \{\cancel{(a, 1)}, \cancel{(b, 2)}, (c, 3), \cancel{(a, 4)}, \cancel{(b, 5)}, (c, 6), (d, 1), (e, 2), (f, 3)\}$

$\{\underline{a}, \underline{b}\} \mathbin{\lhd\mkern-14mu-} r = \{(c,3), (c,6), (d,1), (e,2), (f,3)\}$

$r = \{\cancel{(a, 1)}, \cancel{(b, 2)}, (c, 3), (a, 4), (b, 5), (c, 6), \cancel{(d, 1)}, \cancel{(e, 2)}, (f, 3)\}$

$r \mathbin{\rhd\mkern-14mu-} \{\underline{1}, \underline{2}\} = \{(c,3), (a,4), (b,5), (c,6), (f,3)\}$

$$r = (s \lhd r) \cup (s ⩤ r)$$

Relational Operations: Overriding

$r = \{\cancel{(a, 1)}, (b, 2), \cancel{(c, 3)}, \cancel{(a, 4)}, (b, 5), \cancel{(c, 6)}, (d, 1), (e, 2), (f, 3)\}$

Example: Calculate r overridden with {(a, 3), (c, 4)}

Hint: Decompose results to those in t's domain and those not in t's domain.

$$r \Leftarrow t = \{(d, r') \mid (d, r') \in t \lor ((d, r') \in r \land d \notin \underbrace{}_{dom(t)})\}$$

a relation

$$r \Leftarrow \underbrace{\{(a,3),(c,4)\}}_{t} = \{(d, r') \mid (d, r') \in \{(a,3),(c,4)\} \lor \underbrace{(d,r') \in r \land d \notin \{a, c\}}_{\text{domain subtraction}}\}$$

$$= \{(a,3),(c,4),(b,2),(b,5),(d,1),(e,2),(f,3)\}$$

Problems (don't look at the slides!)

(1) Rewrite the relational image $r[s]$ in terms of dom/ran and/or restrictions/subtractions.

(2) Rewrite the overriding $r \mathbin{\lhd\!\!\!-} t$ in terms of dom/rand and/or restrictions/subtractions and/or set operations.

Lecture 1b

Review on Math: Functions

Functional Property

relation

$$\text{isFunctional}(r) \iff$$

$\to \in S \leftrightarrow T$

$\in S \quad \in T$

$$\forall s, t1, t2 \bullet$$

$$(s \in S \land t1 \in T \land t2 \in T)$$

$$\Rightarrow$$

$$(\boxed{(s, t1) \in r \land (s, t2) \in r \Rightarrow t1 = t2})$$

each domain value s maps to ~~at most one~~ range value

$\{\underset{s\ t1}{(a, 1)}, (b, 2), \underset{s\ t2}{(a, 3)}\}$

$\hookrightarrow$ a relation

not a function!

Q: Smallest relation satisfying the functional property.

Q: How to **prove** or **disprove** that a relation r is a function.

Q: Rewrite the functional property using **contrapositive**.